ON A PROBLEM OF HEAT CONDUCTION IN JETS

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A two-dimensional problem of cooling of a jet with boundaries at prescribed temperature is considered. The problem of cooling of a jet with solid boundaries heated and free boundaries at a prescribed temperature is solved by the Wiener-Hopf method.

1. Let us consider the discharge of a fluid through a longitudinal slot in the boundary of a cylinder, the normal section of which is represented by the line L (Fig.1). The problem of the cooling of a jet discharging through the slot AB reduces to the integration of a heat conduction equation

$$a(\partial^2 T/\partial x^2 + \partial^2 T/\partial y^2) = \mathbf{v}\cdot\operatorname{grad} T \qquad (1.1)$$

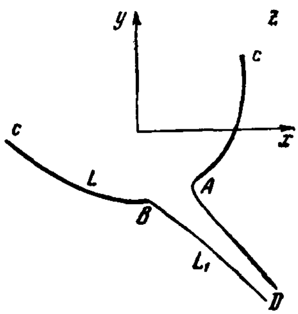

Fig. 1

where a is the coefficient of thermal diffusivity, $\mathbf{v}$ is the velocity vector. The boundary conditions are

$$T = \Theta \text{ on } L + L_1 \qquad (1.2)$$

$$\lim \Theta = T_1 \quad \text{for } z \to cr, \qquad \lim \Theta = T_0 \quad \text{for } z \to D \qquad (z = x + iy)$$

where L_1 denotes the free boundary of the jet.

For simplicity we shall assume that boundary points laying on the same equipotential line are at the same temperature.

Let $w = \varphi + i\psi$ be the complex flow potential function, where φ is the velocity potential and ψ is the stream function. By changing the variables in Equation (1.1) according to $x = x(\varphi,\psi)$, $y = y(\varphi,\psi)$ we shall obtain

$$a(\partial^2 T/\partial\varphi^2 + \partial^2 T/\partial\psi^2) = \partial T/\partial\varphi \qquad (1.3)$$

The region of flow in the φ, ψ plane is limited to the rectangle $|\psi|<\psi_0$ (Fig.2), where the fluid discharge is $2\psi_0$. The boundary conditions expressed by (1.2) for Equation (1.3) assume the form

$$T = \Theta \quad \text{for } |\psi| = \psi_0, \qquad \lim \Theta = \begin{cases} T_1 & (\varphi \to -\infty) \\ T_0 & (\varphi \to +\infty) \end{cases} \qquad (1.4)$$

It is known from the Fourier integral theory that for a function $F(\tau)$ satisfying the Dirchlet conditions on the τ-axis the following equations are true [1]:

$$\lim \frac{1}{\pi i} \int_{-\infty}^{\infty} F(\tau) e^{-i\tau\varphi} \frac{d\tau}{\tau} = \mp F(\pm 0) \quad , \text{ for } \varphi \to \pm\infty \tag{1.5}$$

where the upper sign is taken for $\varphi > 0$, and the lower sign for $\varphi < 0$.

Let us note that for $F = 1$ Equation (1.5) assumes the form

$$\frac{1}{\pi i} \int_{-\infty}^{\infty} e^{-i\tau\varphi} \frac{d\tau}{\tau} = \begin{cases} -1 & (\varphi > 0) \\ 1 & (\varphi < 0) \end{cases} \tag{1.6}$$

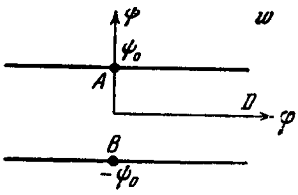

Fig. 2

We shall seek the solution of Equation (1.3) with the help of Equation (1.5) in the form that will satisfy conditions (1.4)

$$T = \frac{T_1 - T_0}{2\pi i} \int_{-\infty}^{\infty} F(\tau, \psi) e^{-i\tau\varphi} \frac{d\tau}{\tau} + \frac{T_1 + T_0}{2} , \tag{1.7}$$

$$F(\pm 0, \psi) = 1$$

Substituting Equation (1.7) into (1.3) for a function of $F(\tau, \psi)$ we shall obtain

$$\frac{d^2 F}{d\psi^2} - \nu^2 F = 0, \qquad \nu = \left(\tau^2 - \frac{i\tau}{a}\right)^{1/2}$$

with which is associated a symmetrical form with respect to ψ

$$F(\tau, \psi) = A(\tau) \frac{\cosh \nu\psi}{\cosh \nu\psi_0}, \qquad A(\pm 0) = 1$$

In this way we find that

$$T = \frac{T_1 - T_0}{2\pi i} \int_{-\infty}^{\infty} A(\tau) \frac{\cosh \nu\psi}{\cosh \nu\psi_0} e^{-i\tau\varphi} \frac{d\tau}{\tau} + \frac{T_1 + T_0}{2} \tag{1.8}$$

From Equation (1.8) and the boundary conditions (1.4) we obtain an integral equation for the function $A(\tau)$

$$\Theta(\varphi) = \frac{T_1 - T_0}{2\pi i} \int_{-\infty}^{\infty} A(\tau) e^{-i\tau\varphi} \frac{d\tau}{\tau} + \frac{T_1 + T_0}{2} \tag{1.9}$$

We seek the solution of this equation in the form of the Fourier integral

$$A(\tau) = \int_{-\infty}^{\infty} B(\lambda) e^{i\tau\lambda} d\lambda, \qquad \int_{-\infty}^{\infty} B(\lambda) d\lambda = 1 \tag{1.10}$$

With the help of Equations (1.10), (1.9) and (1.6) we find (*)

$$\Theta(\varphi) = -\frac{T_1 - T_0}{2}\left[\int_{-\infty}^{\varphi} B(\lambda) d\lambda + \int_{\infty}^{\varphi} B(\lambda) d\lambda\right] + \frac{T_1 + T_0}{2}$$

Hence

$$B(\lambda) = -\frac{1}{T_1 - T_0} \frac{d\Theta(\lambda)}{d\lambda} \tag{1.11}$$

We shall note that the normalizing equation (1.10) for the function $B(\lambda)$ is automatically satisfied.

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*) Here and in the following the interchangeability of the order of integration is assumed.

From Equation (1.8) and with the help of Equations (1.11) and (1.10) we obtain the final solution

$$T = \int_{-\infty}^{\infty} d\Theta / d\lambda \, g(\lambda - \varphi, \psi)\, d\lambda + \frac{T_1 + T_0}{2} \qquad \left(g = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\cosh \nu\psi}{\cosh \nu\psi_0} e^{i\tau(\lambda-\varphi)} \frac{d\tau}{\tau}\right) \tag{1.12}$$

In order to evaluate the integral (1.12) approximately, we shall consider the mean temperature of the stream (*)

$$T^* = \frac{1}{2\psi_0} \int_{-\psi_0}^{\psi_0} T\, d\psi = \int_{-\infty}^{\infty} \frac{d\Theta}{d\lambda} g^*(\lambda - \varphi)\, d\lambda + \frac{T_1 + T_0}{2} \tag{1.13}$$

$$g^*(\lambda - \varphi) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \tanh \nu\psi_0 e^{i\tau(\lambda-\varphi)} \frac{d\tau}{\nu\tau} \tag{1.14}$$

Moreover, using the known development

$$\frac{1}{\nu} \tanh \nu\psi_0 = \frac{2}{\psi_0} \sum_{n=1}^{\infty} \frac{1}{\nu^2 + \mu_n^2} \qquad \left(\mu_n = \frac{\pi}{2\psi_0}(2n-1)\right)$$

we obtain

$$g^*(\lambda - \varphi) = -\frac{1}{\psi_0^2} \sum_{n=1}^{\infty} g_n^*(\lambda - \varphi), \qquad g_n^*(\lambda - \varphi) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{i\tau(\lambda-\varphi)}}{\tau(\nu^2 + \mu_n^2)} d\tau \tag{1.15}$$

The determinator of the integrant of Equation (1.15) has the following roots

$$\tau_0 = 0, \qquad \tau_{1n} = \frac{i}{2a}\left(\sqrt{1 + 4\mu_n^2 a^2} + 1\right)$$

$$\tau_{2n} = -\frac{i}{2a}\left(\sqrt{1 + 4\mu_n^2 a^2} - 1\right)$$

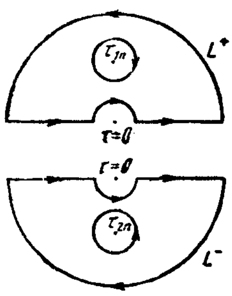

Fig. 3

Therefore, the evaluation of the integrals (1.15) involves consideration of the line integrals (with $\lambda - \varphi > 0$, and $\lambda - \varphi < 0$, respectively)

$$G_n^+ = \frac{1}{\pi i} \oint_{L+} \frac{e^{i\tau(\lambda-\varphi)}}{\tau(\nu^2 + \mu_n^2)} d\tau$$

$$G_n^- = \frac{1}{\pi i} \oint_{L-} \frac{e^{i\tau(\lambda-\varphi)}}{\tau(\nu^2 + \mu_n^2)} d\tau \tag{1.16}$$

The respective paths of integration are shown in Fig.3.

From Equations (1.16), on the basis of the residue theorem and Jordan's lemma we find

$$g_n^*(\lambda - \varphi) = \frac{1}{\mu_n^2} - \frac{4a^2 e^{-|\tau_{1n}|(\lambda-\varphi)}}{\left(\sqrt{1 + 4\mu_n^2 a^2} + 1\right)\sqrt{1 + 4\mu_n^2 a^2}} \qquad \text{for } \lambda - \varphi > 0 \tag{1.17}$$

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*) The mean temperature of the stream is defined with respect to the energy flow through the jet section referred to the energy of a jet of unit temperature flowing through this section with the mean velocity. For sections sufficiently far from the slot the above defined mean temperature becomes identical with the mean temperature for the section.

$$g^*(\lambda-\varphi) = -\frac{1}{\mu_n^2} + \frac{4a^2 e^{|\tau_{2n}|(\lambda-\varphi)}}{(\sqrt{1+4\mu_n^2a^2}-1)\sqrt{1+4\mu_n^2a^2}} \quad \text{for } \lambda-\varphi<0 \quad (1.18)$$

With the help of Formulas (1.17) and (1.18), Equations (1.15) and (1.13) give

$$T^*(\varphi) = \Theta(\varphi) - \frac{4a^2}{\psi_0^2}\sum_{n=1}^{\infty}\frac{1}{\sqrt{1+4\mu_n^2a^2}}\left\{\frac{1}{\sqrt{1+4\mu_n^2a^2}-1}\int_{-\infty}^{\varphi}\frac{d\Theta}{d\lambda}e^{|\tau_{2n}|(\lambda-\varphi)}d\lambda + \right.$$

$$\left. + \frac{1}{\sqrt{1+4\mu_n^2a^2}+1}\int_{\varphi}^{\infty}\frac{d\Theta}{d\lambda}e^{-|\tau_{1n}|(\lambda-\varphi)}d\lambda\right\} \quad (1.19)$$

As an example we shall consider the eflux of fluid from an infinite rectangular slot in a plane. Let L denote the boundary of the upper half-plane $|x| > l$, where $2l$ is the width of the slot. We shall assume that the half-plane constituting the boundary is maintained at a steady temperature T_1, and the jet boundary L_1 at a steady temperature T_0. For this case $d\Theta/d\lambda$ can be represented by the delta-function

$$\frac{d\Theta}{d\lambda} = -(T_1 - T_0)\,\delta(\lambda)$$

and the mean temperature of the jet $(\varphi > 0)$

$$T^*(\varphi) = T_0 + \frac{4a^2(T_1-T_0)}{\psi_0^2}\sum_{n=1}^{\infty}\frac{e^{-|\tau_{2n}|\varphi}}{(\sqrt{1+4\mu_n^2a^2}-1)\sqrt{1+4\mu^2\mu_n^2}} \quad (1.20)$$

In order to determine the relationship between φ and y, we shall use the following expressions, known from the theory of jets [2]

$$W = -\frac{2\psi_0}{\pi}\ln\zeta + i\psi_0 \qquad (\zeta = \xi + i\eta)$$

$$z = \frac{2l}{2+\pi}\left(\zeta + \sqrt{\zeta^2-1} - i\ln\frac{1-i\sqrt{\zeta^2-1}}{\zeta}\right) + \frac{l\pi}{2+\pi}$$

Here the region ζ corresponds to the upper half-plane, the interval $|\xi| < 1$ corresponds to the boundaries of jet and $|\xi| > 1$ correspond to lines $|x| > l$, $y = 0$. With this, the points $\xi = \pm 1$ correspond to the points $x = \pm l$ and $\zeta = 0$ corresponds to a point on the jet at infinity, $x = 0$, $y = -\infty$.

Along the free boundary of the jet the relationship between φ and y is given by Equations

$$\varphi = -\frac{2\psi_0}{\pi}\ln|\xi|, \qquad y = \frac{2l}{2+\pi}\left(\sqrt{1-\xi^2} - \ln\frac{1+\sqrt{1-\xi^2}}{|\xi|}\right), \qquad |\xi| \leqslant 1 \quad (1.21)$$

For sufficiently large distances from the slot the second of Equations (1.21) can be replaced by the apptoximation

$$y = \frac{2l}{2+\pi}\left(1 - \ln\frac{2}{|\xi|}\right) \quad (1.22)$$

We shall note that the approximation (1.22) can also be used for small distances from the slot. E.g. for $y/2l = -0.18$, the error committed by taking the approximate instead of the exact equation (1.21) does not exceed 2.2%.

From Expressions (1.20) to (1.22) the solution is obtained in the form

$$\frac{T^*(y)-T_0}{T_1-T_0} = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\exp\{(\sqrt{\alpha^2+(2n-1)^2}-\alpha)\,[(2+\pi)\,y\,/\,2l + \ln 2 - 1]\}}{(\sqrt{\alpha^2+(2n-1)^2}-\alpha)\sqrt{\alpha^2+(2n-1)^2}} \quad (1.23)$$

where $\alpha = \psi_0/\pi a$.

Equation (1.23) shows that the relative cooling of jets is described by one parameter α , and the cooling effect rapidly decreases with increasing α .

2. Let $w = \varphi + i\psi$ be a known complex potential function for the jet flow of a fluid. Then, the heat conduction equation in the w-plane has the following form

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{\partial T}{\partial x} \qquad \left(x = \frac{\varphi}{a}, \; y = \frac{\psi}{a}\right) \tag{2.1}$$

where a is the coefficient of thermal diffusivity.

Moreover, we shall assume that the flow region corresponds to the strip $|\psi| \leqslant \psi_0$ in the w-plane (defined by $|y| \leqslant y_0$, $y_0 = \psi_0/a$, where $2\psi_0$ is the fluid discharge), and that points of the jet boundary laying on the same equipotential line are at the same temperature

$$T(x, y_0) = T(x, -y_0) \tag{2.2}$$

We stipulate that heat inflow takes place along the solid boundaries of the jet $(x < 0)$ and the free boundaries $(x > 0)$ are at a given temperature. According to Equation (2.2) we have the following boundary conditions for Equation (2.1)

$$T = T_0(x) \quad (x > 0), \qquad \partial T / \partial y = \pm Q_0(x) \quad (x < 0) \qquad \text{for } y = \pm y_0 \tag{2.3}$$

We assume that functions T_0 and Q_0 are integrable in their respective regions and satisfy the following inequalities

$$\begin{aligned} |T_0| &< M \exp(\tau_+ x) \quad \text{for } x \to +\infty \quad (\tau_+ < 0) \\ |Q_0| &< N \exp(\tau_- x) \quad \text{for } x \to -\infty \quad (\tau_- > 0) \end{aligned} \tag{2.4}$$

In order to obtain the solution to the stated problem we shall use the Fourier transforms

$$\Phi(\lambda, y) = \int_{-\infty}^{\infty} T(x, y)\, e^{i\lambda x}\, dx, \qquad T(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi(\lambda, y)\, e^{-i\lambda x} d\lambda \tag{2.5}$$

Let us represent the function Φ by the sum

$$\Phi(\lambda, y) = \Phi_+(\lambda, y) + \Phi_-(\lambda, y) \tag{2.6}$$

where

$$\Phi_+(\lambda y) = \int_0^{\infty} T(x, y)\, e^{i\lambda x} dx, \qquad \Phi_-(\lambda, y) = \int_{-\infty}^{0} T(x, y)\, e^{i\lambda x}\, dx \tag{2.7}$$

With the help of transformations (2.5) the system of equations (2.1) through (2.4) are written in the following form

$$\frac{d^2\Phi}{dy^2} - \gamma^2 \Phi = 0 \quad (\gamma = \sqrt{\lambda^2 - i\lambda}), \qquad \Phi(\lambda, y_0) = \Phi(\lambda, -y_0) \tag{2.8}$$

at $y = \pm y_0$

$$\Phi_+ = f_+(\lambda) = \int_0^{\infty} T_0(x)\, e^{i\lambda x}\, dx, \qquad \frac{d\Phi_-}{dy} = \pm g_-(\lambda) = \pm \int_{-\infty}^{0} Q_0(x)\, e^{i\lambda x}\, dx \tag{2.9}$$

Since Φ can be expressed as a function of y , the general solution of Equation (2.8) is

$$\Phi(\lambda, y) = A(\lambda) \frac{\cosh \gamma y}{\cosh \gamma y_0}$$

Satisfying the boundary condition (2.9) we obtain

$$\Phi_-(\lambda, y_0) + f_+(\lambda) = A(\lambda), \qquad \Psi_+(\lambda, y_0) + g_-(\lambda) = \gamma A(\lambda) \tanh \gamma y_0 \tag{2.10}$$

where

$$\Psi_+(\lambda, y_0) = \frac{d\Phi_+}{dy} \quad \text{for} \quad y = y_0$$

By eliminating the quantity $A(\lambda)$ from Equation (2.10) we arrive at the relationship

$$\gamma \tanh \gamma y_0 \Phi_-(\lambda, y_0) - \Psi_+(\lambda, y_0) + \gamma \tanh \gamma y_0 f_+(\lambda) - g_-(\lambda) = 0 \qquad (2.11)$$

Further, if the parameter λ is taken as a complex quantity $(\lambda = \sigma + i\tau)$, then according to the imposed conditions (2.4) the function with the subscript "plus" are regular at $\tau > \tau_+$, and those with subscript "minus" are regular [1] at $\tau < \tau_-$.

We shall obtain the solution of the boundary value problem (2.11) by the Wiener-Hopf method [3]. Expressing the hyperbolic tangent by the infinite product

$$\gamma \tanh \gamma y_0 = \gamma^2 y_0 \prod_{k=1}^{\infty} \left(\frac{2k-1}{2k}\right)^2 \frac{\gamma^2 + \beta_k^2}{\gamma^2 + \varepsilon_k^2}$$

$$\beta_k = k\alpha, \qquad \varepsilon_k = \left(k - \frac{1}{2}\right)\alpha, \qquad \alpha = \frac{\pi}{y_0} \qquad (2.12)$$

and factorizing

$$\gamma \tanh \gamma y_0 = K_+(\lambda) K_-(\lambda) \qquad (2.13)$$

$$K_+(\lambda) = y_0 \lambda \prod_{k=1}^{\infty} \frac{2k-1}{2k} \frac{\lambda - \lambda_{1k}}{\lambda - \lambda_{3k}}, \qquad K_-(\lambda) = (\lambda - i) \prod_{k=1}^{\infty} \frac{2k-1}{2k} \frac{\lambda - \lambda_{0k}}{\lambda - \lambda_{2k}} \qquad (2.14)$$

$$\lambda_{0k} = {}^1/_2\, i\left(\sqrt{1 + 4k^2\alpha^2} + 1\right), \qquad \lambda_{2k} = {}^1/_2\, i\left(\sqrt{1 + (2k-1)^2\alpha^2} + 1\right)$$

$$\lambda_{1k} = -{}^1/_2\, i\left(\sqrt{1 + 4k^2\alpha^2} - 1\right), \qquad \lambda_{3k} = -{}^1/_2\, i\left(\sqrt{1 + (2k-1)^2\alpha^2} - 1\right) \qquad (2.15)$$

Here function K_+ is regular for $\tau > 0$ and K_- is regular for $\tau < 1$.

Let us substitute (2.13) into Equation (2.11) and divide by K_+. We obtain

$$K_-(\lambda)\Phi_-(\lambda, y_0) - \frac{1}{K_+(\lambda)}\Psi_+(\lambda, y_0) + G(\lambda) = 0, \qquad G(\lambda) = K_-(\lambda) f_+(\lambda) - \frac{g_-(\lambda)}{K_+(\lambda)} \qquad (2.16)$$

We shall note that the function $K_- f_+$ is regular on the strip $\tau_+ < \tau < 1$ and function g_-/K_+ is regular on $0 < \tau < \tau_-$. Therefore $G(\lambda)$ is regular for $0 < \tau < \tau_0$, where τ_0 is the smallest of the quantities $\tau = \tau_-$ and $\tau = 1$. As a result of this we write the quantity $G(\lambda)$ in the form of a sum

$$G(\lambda) = G_+(\lambda) + G_-(\lambda) \qquad \left(G_-(\lambda) = -\frac{1}{2\pi i} \int_{i(\tau_0 - 0) - \infty}^{i(\tau_0 - 0) + \infty} \frac{G(\zeta)}{\zeta - \lambda} d\zeta\right) \qquad (2.17)$$

whose components are regular for $\tau > 0$ and $\tau < \tau_0$.

The expression for $G_-(\lambda)$ is easily obtained by evaluating the line integral

$$F(\lambda) = \frac{1}{2\pi i} \oint_{C_+} \frac{f_+(\zeta) K_-(\zeta)}{\zeta - \lambda} d\zeta - \frac{1}{2\pi i} \oint_{C_-} \frac{g_-(\zeta)\, d\zeta}{K_+(\zeta)(\zeta - \lambda)} \qquad (2.18)$$

Here the path of integration C_+ is along the line $\tau = \tau_0 - 0$ and upper semi-circular arc of infinite radius, and the curve C_- consists of the line $\tau = \tau_0 - 0$ and the lower semi-circular arc of infinite radius. We note that C_+ encloses the poles of the function $K_-(\lambda)$ and C_-, the zeros of the function $K_+(\lambda)$ and the point $\zeta = \lambda$. Substituting into Equation (2.18) the expressions for the functions K_+ and K_- from (2.14) and applying the residue theorem, we obtain

$$G_-(\lambda) = -\sum_{p=1}^{\infty} \frac{2p-1}{2p} \frac{f_+(\lambda_{2p})(\lambda_{2p}-i)(\lambda_{2p}-\lambda_{0p})}{\lambda_{2p}-\lambda} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{2k-1}{2k} \frac{\lambda_{2p}-\lambda_{0k}}{\lambda_{2p}-\lambda_{2k}} -$$

$$-\frac{g_-(0)\,\alpha}{\pi\lambda} \prod_{k=1}^{\infty} \frac{2k}{2k-1} \frac{\lambda_{3k}}{\lambda_{1k}} + \frac{g_-(\lambda)}{K_+(\lambda)} +$$

$$+\frac{\alpha}{\pi} \sum_{p=1}^{\infty} \frac{2p}{2p-1} \frac{g_-(\lambda_{1p})(\lambda_{1p}-\lambda_{3p})}{\lambda_{1p}(\lambda_{1p}-\lambda)} \prod_{\substack{k=1\\k\neq p}}^{\infty} \frac{2k}{2k-1} \frac{\lambda_{1p}-\lambda_{3k}}{\lambda_{1p}-\lambda_{1k}} \qquad (2.19)$$

On the basis of Liouville's theorem the relationships (2.16) and (2.17) give

$$K_-(\lambda)\,\Phi_-(\lambda, y_0) + G_-(\lambda) = 0 \qquad (2.20)$$

where the first term of Equation (2.20) tends to zero as $\lambda \to \infty$. Substituting for Φ_- into the first of equations (2.10), we obtain

$$A(\lambda) = f_+(\lambda) - \frac{G_-(\lambda)}{K_-(\lambda)}$$

Therefore

$$T(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[f_+(\lambda) - \frac{G_-(\lambda)}{K_-(\lambda)}\right] \frac{\cosh \gamma y}{\cosh \gamma y_0} \exp(-i\lambda x)\, d\lambda \qquad (2.21)$$

In order to compute the quantity T approximately, we shall consider the mean temperature of the stream

$$T^*(x) = \frac{1}{2y_0} \int_{-y_0}^{y_0} T(x, y)\, dy =$$

$$= \frac{\alpha}{2\pi^2} \int_{-\infty}^{\infty} f_+(\lambda) \frac{\tanh \gamma y_0}{\gamma} e^{-i\lambda y}\, d\lambda - \frac{\alpha}{2\pi^2} \int_{-\infty}^{\infty} \frac{G_-(\lambda)}{K_-(\lambda)} \frac{\tanh \gamma y_0}{\gamma} e^{-i\lambda x}\, d\lambda \qquad (2.22)$$

and substitute into this equation the expression for $\tanh \gamma y_0$ from Equation (2.13)

$$T^*(x) = \frac{\alpha}{2\pi^2} \int_{-\infty}^{\infty} f_+(\lambda) \frac{\tanh \gamma y_0}{\gamma} e^{-i\lambda x}\, d\lambda + T_0^*(x) \qquad (2.23)$$

$$T_0^*(x) = -\frac{\alpha}{2\pi^2} \int_{-\infty}^{\infty} \frac{1}{\gamma^2} G_-(\lambda) K_+(\lambda) e^{-i\lambda x}\, d\lambda \qquad (2.24)$$

We observe that the quantity $\alpha = \pi a / \psi_0 \ll 1$ holds also for relatively small fluid discharges. Therefore all quantities shall be estimated by comparing them with respective powers of α.

We are investigating the temperature of the jet $(x > 0)$. For $x > 0$ the integral (2.24) can be easily evaluated. Accordingly, for $x > 0$ we can consider the line integral

$$\frac{1}{2\pi i} \oint_C \frac{1}{\gamma^2} G_-(\lambda) K_+(\lambda) \exp(-i\lambda x)\, d\lambda$$

where the curve C consists of the line $\tau = 0$ and the lower semi-circular arc of infinitely large radius. Enclosed by this curve are only the poles of the function K_+. Using Expression (2.14) for the function K_+ and with the help of the residue theorem we get

$$T_0^* = -i\sum_{n=1}^{\infty}\frac{2n-1}{2n}\frac{G_-(\lambda_{3n})(\lambda_{3n}-\lambda_{1n})}{\lambda_{3n}-i}e^{-i\lambda_{3n}x}\prod_{\substack{k=1\\k\neq n}}^{\infty}\frac{2k-1}{2k}\frac{\lambda_{3n}-\lambda_{1k}}{\lambda_{3n}-\lambda_{3k}} \tag{2.25}$$

The infinite products entering Equations (2.25) and (2.19) can be transformed as follows. From relationships (2.12) and (2.14) we have

$$\prod_{\substack{k=1\\k\neq p}}^{\infty}\frac{2k-1}{2k}\frac{\lambda_{2p}-\lambda_{0k}}{\lambda_{2p}-\lambda_{2k}} = \tag{2.26}$$

$$= \frac{2p\alpha}{\pi(2p-1)(\lambda_{2p}-\lambda_{0p})\gamma_{2p}}\prod_{k=1}^{\infty}\frac{2k}{2k-1}\frac{\lambda_{2p}-\lambda_{3k}}{\lambda_{2p}-\lambda_{1k}}\lim\,(\lambda-\lambda_{2p})^{\tanh\gamma y_0}$$

$$\text{for } \lambda\to\lambda_{2p} \qquad (\gamma_{2p}=\gamma(\lambda_{2p}))$$

Evaluating the limit we obtain

$$\prod_{\substack{k=1\\k\neq p}}^{\infty}\frac{2k-1}{2k}\frac{\lambda_{2p}-\lambda_{0k}}{\lambda_{2p}-\lambda_{2k}} = \frac{4p\alpha^2\Pi_{2p}}{\pi^2(2p-1)(\lambda_{2p}-\lambda_{0p})(2\lambda_{2p}-i)}$$

$$\Pi_{2p} = \prod_{k=1}^{\infty}\frac{2k}{2k-1}\frac{\lambda_{2p}-\lambda_{3k}}{\lambda_{2p}-\lambda_{1k}} \tag{2.27}$$

Similarly, the other infinite products transform to

$$\prod_{\substack{k=1\\k\neq p}}^{\infty}\frac{2k}{2k-1}\frac{\lambda_{1p}-\lambda_{3k}}{\lambda_{1p}-\lambda_{1k}} = \frac{(2p-1)\gamma_{1p}^2\Pi_{1p}}{p(2\lambda_{1p}-i)(\lambda_{1p}-\lambda_{3p})}, \qquad \Pi_{1p} = \prod_{k=1}^{\infty}\frac{2k-1}{2k}\frac{\lambda_{1p}-\lambda_{0k}}{\lambda_{1p}-\lambda_{2k}}$$

$$\prod_{\substack{k=1\\k\neq p}}^{\infty}\frac{2k-1}{2k}\frac{\lambda_{3n}-\lambda_{1k}}{\lambda_{3n}-\lambda_{3k}} = \frac{4n\alpha^2\Pi_{3n}}{\pi^2(2n-1)(\lambda_{3n}-\lambda_{1n})(2\lambda_{3n}-i)} \tag{2.28}$$

$$\Pi_{3n} = \prod_{k=1}^{\infty}\frac{2k}{2k-1}\frac{\lambda_{3n}-\lambda_{2k}}{\lambda_{3p}-\lambda_{0k}} \tag{2.29}$$

The infinite products (2.27) and (2.28) have the following approximations (*)

$$\Pi_{2p} = \frac{4}{l}\left(\frac{2(1+a_p)}{\alpha l}\right)^{1/2}, \qquad \Pi_{1p} = \frac{l}{4}\left(\frac{\alpha l}{2(1+b_p)}\right)^{1/2}, \qquad \Pi_{3n} = \frac{4}{l}\left(\frac{2(1+a_n)}{\alpha l}\right)^{1/2} \tag{2.30}$$

$$\left(a_n = \sqrt{1+(2n-1)^2\alpha^2},\ b_n = \sqrt{1+4n^2\alpha^2}\right)$$

Moreover

$$\prod_{k=1}^{\infty}\frac{2k}{2k-1}\frac{\lambda_{3k}}{\lambda_{1k}} = \frac{1}{8l}\left(\frac{\alpha}{l}\right)^{1/2} \tag{2.31}$$

*) The methods of approximating expressions of the type (2.27) through (2.29) are considered more fully in Section 3.

Using relationships (2.30) and (2.31) from Equations (2.25), (2.19), (2.27) to (2.29) and (2.15) we get

$$T_0^* = -\frac{2^8\alpha^3}{\pi^4 l^3}\sum_{n,p=1}^{\infty} A_{n,p} + \frac{4\sqrt{2}\alpha^3 g_-(0)}{\pi^3 l^3}\sum_{n=1}^{\infty}\frac{\exp[-{}^1/_2 x(a_n-1)]}{a_n(a_n-1)\sqrt{a_n+1}} + \frac{8\alpha^3}{\pi^3}\sum_{n,p=1}^{\infty} B_{n,p} \tag{2.32}$$

where

$$A_{n,p} = \frac{f_+(\lambda_{2p})(a_p-1)\sqrt{a_p+1}}{a_n a_p\sqrt{a_n+1}(a_n+a_p)}\exp\left[-\frac{x}{2}(a_n-1)\right]$$

$$B_{n,p} = \frac{g_-(\lambda_{1p})\sqrt{b_p+1}}{b_p a_n\sqrt{a_n+1}(a_n-b_p)}\exp\left[-\frac{x}{2}(a_n-1)\right] \tag{2.33}$$

The convergence of the single series of expression (2.32) is obvious. The double series are also convergent as the integrals

$$\int_1^{\infty}\int_1^{\infty} A_{\xi,\eta}\,d\xi,\,d\eta, \qquad \int_1^{\infty}\int_1^{\infty} B_{\xi,\eta}\,d\xi\,d\eta$$

and exist nonzero limits [4]

$$\underline{\lim}_{n+p\to\infty}\frac{A_{n+1,\,p+1}}{A_{n,\,p}} = A_0 e^{-x\alpha}, \quad \underline{\lim}_{n+p\to\infty}\frac{B_{n+1,\,p+1}}{B_{n,\,p}} = B_0 e^{-x\alpha} \quad (A_0 \neq 0,\ B_0 \neq 0)$$

(the line under the symbol lim denotes the minimum limit for $n + p \to \infty$ but for arbitraray values of n or p).

Using the Euler's formula for integral estimation of series [4 and 5] an asymptotic behavior of the function T_0^* can be obtained from (2.32). With accuracy corresponding to terms of order α^0 we obtain

$$T_0^*(x) = T^*(x) = -g_-(0)\frac{(6+\sqrt{2})l^3 - 3}{3l^2\pi^3}\frac{\sqrt{\pi}}{\sqrt{x}} + O\left(\frac{1}{x}\right) \tag{2.34}$$

Formula (2.34) shows that the temperature of the jet at large distances is determined mainly by the sum of heat quantities proportional to

$$g_-(0) = \int_{-\infty}^{0} Q_0(x)\,dx$$

and decreases as $x^{-1/2} = (la/2\psi_0\xi)^{1/2}$, where l is the jet thickness at infinity and ξ is the distance of the considered jet section from the discharging aperture.

3. We are giving here some results for the summation of series. Let us consider the series

$$S_{1,m} = \sum_{k=1}^{m} a(k) = \sum_{k=1}^{n-1} a(k) + S_{n,m} \qquad S_{n,m} = \sum_{k=n}^{m} a(k) \qquad (n \leqslant m) \tag{3.1}$$

The terms $a(x)$ are continuous functions of the variable x $(1 \leqslant x \leqslant m+1)$. We shall assume that the quantities $a(k)$ can be expressed by

$$a(k) = A(k + {}^1/_2) - A(k) \tag{3.2}$$

In order to estimate the last term of the series we shall use the well known Simpson's formula

$$\int_n^{m+1} a(x)\,dx \approx \frac{1}{6}\sum_{k=n}^{m}\left[a(k) + 4a\left(k+\frac{1}{2}\right) + a(k+1)\right] \tag{3.3}$$

From Equation (3.3) with the help of (3.2) it is easy to obtain the approximate value $S_{n,m} = S_{n,m}^*$

$$S_{n,m}^* = \frac{1}{2}[a(m+1) - a(n)] + 2[A(m+1) - A(n)] - 3\int_n^{m+1} a(x)\,dx$$

In the particular case where $a(x)$ is integrable over the interval $(1,\infty)$ we have

$$\sum_{k=1}^{\infty} a(k) \approx S_{1,\infty}^{*} = \sum_{k=1}^{n-1} a(k) - \frac{1}{2} a(n) + 2\,[A(\infty) - A(n)] - 3 \int_{n}^{\infty} a(x)\,dx \qquad (3.4)$$

It is obvious, that for a given n the approximation (3.4) becomes better the slower the convergence of the series.

Thus, for example, let

$$S_{1,\infty} = \sum_{k=1}^{\infty} \frac{1}{2k\,(2k+1)} = 1 - \ln 2 = 0.30685, \qquad A(k) = \frac{2k-1}{2k}$$

For $n = 1$ we have $S_{1,\infty}^{*} = 0.30848$, with an error $(S_{1,\infty} - S_{1,\infty}^{*})/S_{1,\infty} = -0.53 \times 10^{-2}$. For $n = 2$ we have $S_{1,\infty}^{*} = 0.30695$, the error in this case being -0.33×10^{-3}.

Equation (3.4) can be readily adapted for the computation of infinite products.

In fact, supposing

$$a(k) = \ln \frac{\alpha(k + {}^1/_2)}{\alpha(k)} \qquad (3.5)$$

from Equation (3.4) we get

$$\Pi = \prod_{k=1}^{\infty} \frac{\alpha(k + {}^1/_2)}{\alpha(k)} \approx \frac{c^3}{\alpha^{3/2}(n)\,\alpha^{1/2}(n + {}^1/_2)} \exp\left[-3 \int_{n}^{\infty} \ln \frac{\alpha(x + {}^1/_2)}{\alpha(x)}\,dx\right] \prod_{k=1}^{n-1} \frac{\alpha(k + {}^1/_2)}{\alpha(k)}$$

with

$$\prod_{k=1}^{0} \frac{\alpha(k + {}^1/_2)}{\alpha(k)} = 1, \qquad c = \lim \frac{\alpha(k + {}^1/_2)}{\alpha(k)} \quad \text{for } k \to \infty$$

For example,

$$\Pi = \prod_{k=1}^{\infty} \frac{2k}{2k-1}\, \frac{\sqrt{1 + (2k-1)^2 \varepsilon^2} + a}{\sqrt{1 + 4k^2\varepsilon^2} + a}$$

$$\alpha(k) = \frac{(2k-1)\,\varepsilon}{\sqrt{1 + (2k-1)^2 \varepsilon^2} + a} \qquad (0 < \varepsilon \ll 1)$$

Substituting for $a(k)$ into Equation (3.5), for $n = 1$ and considering only terms of order $\varepsilon^{3/2}$, we get

$$\Pi = \frac{2^{3/2}}{l^{3/2}\varepsilon^{1/2}} \sqrt{1 + a} \quad \text{for } a > -1, \qquad \Pi = \frac{\varepsilon^{1/2}}{8 l^{3/2}} \quad \text{for } a = -1$$

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